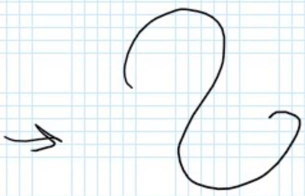
Section 6.4 – Arc Length

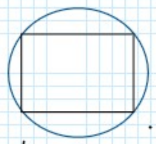
goal:

determine length of this kind of curve



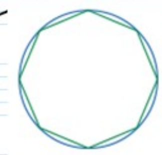
not a function curve (fails VLT)

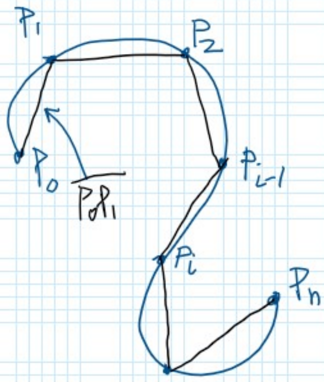
To determine circumference of circle: estimate it using polygon



estimate circumference of circle using perimeter of rectangle

perimeter of octagon gives a better estimate of circle's circumference





curve C

- C is smooth
- f, g 's derivatives are continuous and not zero for $a<t<b$

curve, C, is described by the parametric equations

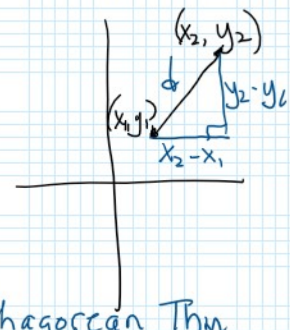
$x = f(t)$, $y = g(t)$ $\quad a \le t \le b$

horizontal movement · time · vertical movement

Of course, the higher the number of straight lines used will result in a higher accuracy of estimate of length of C.

$$L = \text{Length of } C \approx |\overline{P_0P_1}| + |\overline{P_1P_2}| + \dots + |\overline{P_{i-1}P_i}| + \dots + |\overline{P_{n-1}P_n}|$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |\overline{P_{i-1}P_i}| \longleftarrow$$



to develop Arc Length integral, use Distance Formula:

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

by Pythagorean Thm

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\therefore |\overline{P_{i-1}P_i}| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

one line

$x = f(t)$

$$\frac{\Delta x_i}{\Delta t_i} \Leftarrow \frac{dx}{dt} \Leftarrow f'(t)$$

$(ab)^2 = a^2b^2$

since $f'(t_i) = \dfrac{\Delta x_i}{\Delta t_i} \xRightarrow{\text{differential}} \Delta x_i = f'(t_i)\Delta t_i$

likewise: $\Delta y_i = g'(t_i)\Delta t_i$

$$|\overline{P_{i-1}P_i}| = \sqrt{[f'(t_i)\Delta t_i]^2 + [g'(t_i)\Delta t_i]^2}$$

$$= \sqrt{[f'(t_i)]^2(\Delta t_i)^2 + [g'(t_i)]^2(\Delta t_i)^2}$$

$$= \sqrt{(\Delta t_i)^2}\sqrt{(f'(t_i))^2 + (g'(t_i))^2}$$

$$|\overline{P_{i-1}P_i}| = \Delta t_i\sqrt{(f'(t_i))^2 + (g'(t_i))^2}$$

$$L \approx \sum_{i=1}^{n} \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\,\Delta t_i$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

given smooth curve defined by parametric equations

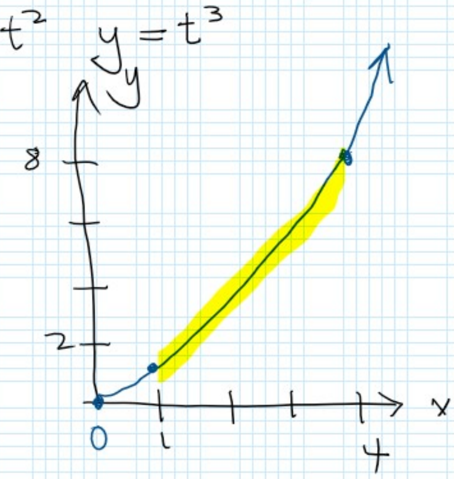
$\longrightarrow x = f(t)$, $y = g(t)$ $a \le t \le b$

curve is traversed exactly once as t increases from a to b

ex. find the length of the arc of a curve $x = t^2$ $y = t^3$ that lies between $(1,1)$ and $(4,8)$

t	x (t^2)	y (t^3)
0	0	0
$t = 1$	1	1
$t = 2$	4	8

integration bounds WRT t



$$L = \int_1^2 \sqrt{[(t^2)']^2 + [(t^3)']^2}\, dt$$
$$= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$
$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$
$$= \int_1^2 \sqrt{t^2(4 + 9t^2)}\, dt$$
$$= \int_{1=t}^{2=t} t\sqrt{4 + 9t^2}\, dt$$

$u = 4 + 9t^2$

$\frac{1}{18}\, du = t\,dt$

$u_{t=1} : 13$

$u_{t=2} : 4 + 9(4) = 40$

$$= \frac{1}{18}\int_{13}^{40} u^{1/2}\, du$$
$$= \frac{1}{\cancel{18}_{9}} \cdot \frac{\cancel{2}}{3} u^{3/2} \Big|_{13}^{40}$$
$$= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right)$$
$$= \frac{1}{27}\left(40\sqrt{40} - 13\sqrt{13}\right)$$

$40\sqrt{4}\sqrt{10}$

$40(2)\sqrt{10}$

$80\sqrt{10}$

Sometimes, equation given is not WRT time.

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \qquad \boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy}$$

ex. setup the length of the arc of the parabola $y^2 = x$ from $(0,0)$ and $(1,1)$

get bounds from y

$x = y^2$ ← integrate wrt y

$\left(\frac{dx}{dy}\right)^2 = (2y)^2 = 4y^2$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$
$$= \int_0^1 \sqrt{4y^2 + 1}\, dy$$

not easily integratable

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt} \qquad \boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy}$$

Do: find exact length of curve for:

$x = 1 + 3t^2 \rightarrow \frac{dx}{dt} = 6t$

$y = 4 + 2t^3 \rightarrow \frac{dy}{dt} = 6t^2$

$\rightarrow 0 \le t \le 1$

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$
$$= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$
$$= \int_0^1 \sqrt{36t^2}\sqrt{(1 + t^2)}\, dt$$

$u = 1 + t^2$

Do: find exact length of curve for:

$x = y^{3/2}$ $(0 \le y \le 1)$

$x = y^{3/2}$

$\left(\frac{dx}{dy}\right)^2 = \left(\frac{3}{2}y^{1/2}\right)^2 = \frac{9}{4}y$

$$\int_0^1 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$
$$= \int_0^1 \sqrt{\frac{9}{4}y + 1}\, dy$$
$$= \frac{4}{9}\int^{13/4} u^{1/2}\, du$$

$u = \frac{9}{4}y + 1$

$= \int_0^1 \sqrt{36t^2}\sqrt{(1+t^2)}\,dt$

$= 6\int_0^1 \cancel{t}\,\sqrt{\underbrace{1+t^2}_{u}}\;t\,dt$

$u = 1+t^2$

$\frac{1}{2}du = t\,dt$

$= 3\int_{u=1}^{u=2} u^{1/2}\,du$

$L = 2(2\sqrt{2}-1)$ or $2^{3/2}$

$= \frac{4}{9}\int_1^{13/4} u^{1/2}\,du$

$u = \frac{9}{4}y + 1$

$\frac{4}{9}du = dy$

$y=0: u_A = 1$

$y=1: u_B = \frac{13}{4}$

$= \frac{4}{9}\cdot\frac{2}{3}\,u^{3/2}\Big|_1^{13/4}$

$= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$

$\frac{13^{3/2}}{4^{3/2}} = \frac{\sqrt{13}\sqrt{13}\sqrt{13}}{8}$

$L = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right)$

ex. find the length of the arc where $x = 4(\cos\theta + \theta\sin\theta)$
$y = 4(\sin\theta - \theta\cos\theta)$

$0 \le \theta \le \pi$

$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$= \int_0^\pi \sqrt{16\theta^2\cos^2\theta + 16\theta^2\sin^2\theta}\,d\theta$

$= \int_0^\pi \sqrt{16\theta^2}\sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{1}}\,d\theta$

$= \int_0^\pi 4\theta\,d\theta$

$= \frac{4}{2}\theta^2\Big|_0^\pi$

$= 2(\pi^2) = \boxed{2\pi^2}$

$x = 4(\cos\theta + \theta\sin\theta)$

$\frac{dx}{d\theta} = 4(-\cancel{\sin\theta} + \cancel{\sin\theta} + \theta\cos\theta)$

$= 4\theta\cos\theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = 16\theta^2\cos^2\theta$

Do: find $\frac{dy}{d\theta}$

$y = 4(\sin\theta - \theta\cos\theta)$

$\frac{dy}{d\theta} = 4(\cos\theta - (\cos\theta \mp \theta\sin\theta))$

$= 4(\cancel{\cos\theta} - \cancel{\cos\theta} + \theta\sin\theta)$

$= 4\theta\sin\theta$

$\left(\frac{dy}{d\theta}\right)^2 = 16\theta^2\sin^2\theta$

$(\theta\sin\theta)'$

$(f\;g)' = f'g + fg'$

$= 1\sin\theta + \theta\cos\theta$

$f = \theta \quad g = \sin\theta$

$f' = 1 \quad g' = \cos\theta$

$(\underset{f}{\theta}\underset{g}{\cos\theta})'$

$\boxed{\begin{matrix} f=\theta & g=\cos\theta \\ f'=1 & g'=-\sin\theta \end{matrix}}$

$y = 4(\sin\theta - \theta\cos\theta)$

$\frac{dy}{d\theta} = 4(\cos\theta - (\cos\theta \mp \theta\sin\theta))$